

QUASI-PRIME SUBMODULES AND DEVELOPED ZARISKI TOPOLOGY

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ABSTRACT. Let R be a commutative ring with nonzero identity and M be an R -module. Quasi-prime submodules of M and the developed Zariski topology on $q\text{Spec}(M)$ are introduced. We also, investigate the relationship between the algebraic properties of M and the topological properties of $q\text{Spec}(M)$. Modules whose developed Zariski topology is respectively T_0 , irreducible or Noetherian are studied, and several characterizations of such modules are given.

1. INTRODUCTION

Prime submodules of modules were introduced as a generalization of prime ideals of rings by J. Dauns [Dau78] and several algebraists carried out an intensive and systematic study of the spectrum of prime submodules (e.g. [Lu84], [MM92], [Lu95], [MMS97], [MMS98], [Lu99], [MS02], [Lu07]). Here, quasi-prime submodules of M as a generalization of prime submodules are introduced. We also, investigate the quasi-primeful modules and we apply them to develop of topological properties of $q\text{Spec}(M)$, where $q\text{Spec}(M)$ is the set of all quasi-prime submodules of M .

The Zariski topology on the spectrum of prime ideals of a ring is one of the main tools in Algebraic Geometry. In the literature, there are many different generalizations of the Zariski topology of rings to modules (see [MMS97], [BH08a], [BH08b], or [Lu99]). In this paper, we are going study the developed Zariski topology as a generalization of the Zariski topology considered in [Lu99], to $q\text{Spec}(M)$, where M is an R -module. As is well known, the Zariski topology has been defined on the set of all prime submodules of a module. Here, we considered developed Zariski topology on the set of all quasi-prime submodules of a module.

Throughout this paper, all rings are commutative with identity and all modules are unital. For a submodule N of an R -module M , $(N :_R M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and annihilator of M , denoted by $\text{Ann}_R(M)$, is the ideal $(0 :_R M)$. M is called faithful if $\text{Ann}(M) = (0)$. If there is no ambiguity we write $(N : M)$ (resp. $\text{Ann}(M)$) instead of $(N :_R M)$ (resp. $\text{Ann}_R(M)$). A proper ideal I of a ring R is said to be *quasi-prime* if for each pair of ideals A and B of R , $A \cap B \subseteq I$ yields either $A \subseteq I$ or $B \subseteq I$ (see [Azi08], [Bou72] and [HRR02]). It is easy to see that every prime ideal is a quasi-prime ideal. Also, every quasi-prime ideal is irreducible (an ideal I of a commutative ring R is said to be *irreducible* if I is not the intersection of two ideals of R that properly contain it).

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A submodule N of an R -module M is said to be *prime* if $N \neq M$ and whenever $rm \in N$ (where $r \in R$ and $m \in M$), then $r \in (N : M)$ or $m \in N$. If N is prime, then the ideal $\mathfrak{p} = (N : M)$ is a prime ideal of R . In this circumstances, N is said to be \mathfrak{p} -*prime* (see [Lu84]). A submodule Q of an R -module M is said to be *primary* if $Q \neq M$ and if $rm \in Q$, where $r \in R$ and $m \in M$ implies that either $m \in Q$ or $r \in \mathfrak{q} = \sqrt{(Q : M)}$. If Q is primary, then $(Q : M)$ is a primary ideal of R . In this case we say that Q is \mathfrak{q} -*primary*, where $\mathfrak{q} = \sqrt{(Q : M)}$ is a prime ideal of R . The set of all prime submodules of an R -module M is called the *prime spectrum* of M and denoted by $\text{Spec}(M)$. Similarly, the collection of all \mathfrak{p} -prime submodules of an R -module M is designated by $\text{Spec}_{\mathfrak{p}}(M)$ for any $\mathfrak{p} \in \text{Spec}(R)$. We remark that $\text{Spec}(\mathbf{0}) = \emptyset$ and that $\text{Spec}(M)$ may be empty for some nonzero module M . For example, the $\mathbb{Z}(p^\infty)$ as a \mathbb{Z} -module has no prime submodule for any prime integer p (see [Lu95]). Such a module is said to be *primeless*.

An R -module M is called *primeful* if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and the map $\Phi : \text{Spec}(M) \rightarrow \text{Spec}(R/\text{Ann}(M))$ defined by $\Phi(P) = (P : M)/\text{Ann}(M)$ for every $P \in \text{Spec}(M)$, is surjective (see [Lu07]). The set of all maximal submodules of an R -module M is denoted by $\text{Max}(M)$. The *Jacobson radical* $\text{Rad}(M)$ of a module M is the intersection of all its maximal submodules. $\text{Rad}(M) = M$ when M has no any maximal submodule. By $N \leq M$ we mean that N is a submodule of M . Let \mathfrak{p} be a prime ideal of R , and $N \leq M$. By the *saturation of N with respect to \mathfrak{p}* , we mean the contraction of $N_{\mathfrak{p}}$ in M and designate it by $S_{\mathfrak{p}}(N)$ and we say N is *saturated with respect to \mathfrak{p}* if $S_{\mathfrak{p}}(N) = N$ (see [Lu03]).

An R -module M is called a *multiplication* module if every submodule N of M is of the form IM for some ideal I of R . For any submodule N of an R -module M we define $V^M(N)$ to be the set of all prime submodules of M containing N . The *radical* of N defined to be the intersection of all prime submodules of M containing N and denoted by $\text{rad}_M(N)$ or briefly $\text{rad}(N)$. $\text{rad}_M(N) = M$ when M has no any prime submodule containing N . In particular, $\text{rad}(\mathbf{0}_M)$ is the intersection of all prime submodules of M . If $V^M(N)$ has at least one minimal member with respect to the inclusion, then every minimal member in this form is called a *minimal prime submodule of N* or a *prime submodule minimal over N* . A minimal prime submodule of $(\mathbf{0})$ is called *minimal prime submodule of M* . A quasi-prime submodule N of an R -module M is called *minimal quasi-prime* if, for any quasi-prime K of M such that $K \subseteq N$, this is the case that $K = N$. An R -module M is said to be *semiprimitive* (resp. *reduced*) if the intersection of all maximal (resp. prime) submodules of M is equal to zero. A submodule N of an R -module M is said *quasi-semiprime* if it is an intersection of quasi-prime submodules. We recall that an R -module M is *co-semisimple* in case every submodule of M is the intersection of maximal submodules (see [AF92, p.122]). Every proper submodule of a co-semisimple module is a quasi-semiprime submodule.

In Section 2, we obtain some properties of quasi-prime submodules. In this section the relations between quasi-prime submodules of a module M and quasi-prime submodules of localizations of M are studied. We also investigate the quasi-primeful modules and we apply them to develop topological properties of $q\text{Spec}(M)$. We show in Theorem 2.14 that an R -module M is quasi-primeful whenever R is a *PID* and M is finitely generated, or R is Laskerian and M is a locally free R -module. We study some main properties of quasi-primeful modules in Proposition 2.16 and also the quasi-prime-embedding modules are studied in Theorem 2.24. It

is shown that an R -module M is top in the cases R is a one dimensional Noetherian domain and either M is weak multiplication or for every prime ideal $\mathfrak{p} \in \text{Spec}(R)$, $|\text{Spec}_{\mathfrak{p}}(M)| \leq 1$ and $S_{(0)}(\mathbf{0}) \subseteq \text{rad}(\mathbf{0})$. In Section 3, we introduce a topology on the set of quasi-prime submodules in such a way that the Zariski topology (see [Lu99]) is a subspace of this topology and some concerned properties are given. An R -module whose developed Zariski topology is T_0 , irreducible or Noetherian is studied in Section 3.

2. SOME PROPERTIES OF QUASI PRIME SUBMODULES

In this section we introduce the notion of quasi-prime submodule and find some properties of it. We also introduce the notions of quasi-primeful and quasi-prime-embedding modules and we use them in the next section.

Definition 2.1. A proper submodule N of an R -module M is called *quasi-prime* if $(N :_R M)$ is a quasi-prime ideal of R .

We define the *quasi-prime spectrum* of an R -module M to be the set of all quasi-prime submodules of M and denote it by $q\text{Spec}^R(M)$. If there is no ambiguity we write only $q\text{Spec}(M)$ instead of $q\text{Spec}^R(M)$. For any $I \in q\text{Spec}(R)$, the collection of all quasi-prime submodules N of M with $(N : M) = I$ is designated by $q\text{Spec}_I(M)$. We say that R is a *serial ring* if the set of all ideals of R is linearly ordered. Recall that a ring R is said to be *arithmetical*, if for any maximal ideal \mathfrak{p} of R , $R_{\mathfrak{p}}$ is a serial ring (see [Jen66]). Recall that a module M is said to be a *Laskerian module*, if every proper submodule of M has a primary decomposition. We know that every Noetherian module is Laskerian.

Remark 2.2. (See [Azi08], [HRR02] and [Jen66]) Let I be an ideal in a ring R and S be a multiplicatively closed subset of R . Then

- (1) If I is quasi-prime, then I is irreducible;
- (2) If R is a Laskerian ring, then every quasi-prime ideal is a primary ideal;
- (3) If I is a prime ideal, then I is quasi-prime;
- (4) Every proper ideal of a serial ring is quasi-prime;
- (5) If IR_S is a quasi-prime ideal of R_S , then $IR_S \cap R$ is a quasi-prime ideal of R ;
- (6) If I is a quasi-prime and primary ideal of R such that $I \cap S = \emptyset$, then IR_S is a quasi-prime ideal of R_S ;
- (7) If R is an arithmetical ring, I is irreducible if and only if I is quasi-prime;
- (8) In an arithmetical ring R any primary ideal is irreducible;
- (9) If R is a Dedekind domain, then I is quasi-prime if and only if I is a primary ideal.

Remark 2.3. Let M be an R -module.

- (1) By [Lu84, Proposition 4], every maximal submodule of an R -module M is prime and by Remark 2.2, every prime submodule of M is a quasi-prime submodule. Therefore, $\text{Max}(M) \subseteq \text{Spec}(M) \subseteq q\text{Spec}(M)$. So, $q\text{Spec}(M) \neq \emptyset$ if M is not primeless.
- (2) Consider $M = \mathbb{Z} \oplus \mathbb{Z}$ as a \mathbb{Z} -module and $N = (2, 0)\mathbb{Z}$ is the submodule of M generated by $(2, 0) \in M$. Then $(N : M) = (0) \in \text{Spec}(\mathbb{Z})$, i.e., $N \in q\text{Spec}(M)$ though N is not a (0) -prime submodule of M . Thus in

general, a quasi-prime submodule need not be a prime submodule, i.e., $\text{Spec}(M) \neq q\text{Spec}(M)$.

- (3) As another example, we consider the faithful torsion \mathbb{Z} -module $M = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$, where p runs through the set of all prime integers. Let $N = (\mathbf{0})$ and $\mathfrak{p} = (0)$. Then $(N : M) = (\mathbf{0} : M) = \text{Ann}(M) = (0)$. Hence, $N \in q\text{Spec}(M)$. However, N is not a prime submodule by [Lu03, Result 2], because $S_{\mathfrak{p}}(N) = S_{(0)}(\mathbf{0}) = M$.

An R -module M is called a *fully prime module* if every proper submodule is a prime submodule. In [BKK04, Proposition 1.10], the authors give several equivalent conditions for an R -module M to be fully prime, for example, M is a fully prime R -module if and only if $\text{Ann}(M)$ is a maximal ideal, i.e., if and only if M is a homogeneous semisimple module (i.e., a direct sum of isomorphic simple R -modules).

Lemma 2.4. *Let $J \in q\text{Spec}(R)$, $\mathfrak{p} \in \text{Spec}(R)$, I be a proper ideal of R and M be an R -module with submodule N . Let S be a multiplicatively closed subset of R .*

- (1) *If $N \in q\text{Spec}_J(M)$, then $(N : M)M \in q\text{Spec}_J(M)$;*
- (2) *If $\{N_\lambda\}_{\lambda \in \Lambda}$ is a family of quasi-prime submodules with $(N_\lambda : M) = J$ for each $\lambda \in \Lambda$, then $\bigcap_{\lambda \in \Lambda} N_\lambda \in q\text{Spec}_J(M)$;*
- (3) *If M is a fully prime module, then every proper submodule of M is quasi-prime. In particular, every proper subspace of a vector space over a field is quasi-prime;*
- (4) *If R is a serial ring, then every proper submodule of M is quasi-prime;*
- (5) *Let N be a quasi-prime submodule of the R_S -module M_S . Then $N \cap M$ is a quasi-prime submodule of M . So, $\{N \cap M \mid N \in q\text{Spec}(M_S)\} \subseteq q\text{Spec}(M)$;*
- (6) *Let R be Laskerian and M be a finitely generated R -module. If N is a quasi-prime submodule of M and $\sqrt{(N : M)} \cap S = \emptyset$, then N_S is a quasi-prime submodule of M_S ;*
- (7) *Let R be an arithmetical ring. Then every primary submodule of M is quasi-prime;*
- (8) *Let R be an arithmetical ring. If $\mathfrak{p} \in V^R(I)$, then $S_{\mathfrak{p}}(I)$ is a quasi-prime ideal of R . Moreover, if R is Laskerian, then $S_{\mathfrak{p}}(I)$ is primary and \mathfrak{p} is a minimal prime ideal over I ;*
- (9) *Let R be an arithmetical ring. Let N be a submodule of M and $\mathfrak{p} \in \text{Supp}(M/N)$. Then $S_{\mathfrak{p}}(N)$ is a quasi-prime submodule of M . Therefore, every proper saturated submodule N w.r.t \mathfrak{p} , is a quasi-prime submodule of M ;*
- (10) *Let R be an arithmetical ring and let M be a finitely generated R -module. If N is a quasi-prime submodule of M and $\mathfrak{p} \in V^R(N : M)$, then $N_{\mathfrak{p}}$ is a quasi-prime submodule of $M_{\mathfrak{p}}$.*

Proof. (1)-(3) are clear.

- (4) Every proper ideal of R is quasi-prime by Remark 2.2.
- (5) One can obtain that $((N \cap M) :_R M) = (N :_{R_S} M_S) \cap R$. Now, let $I := (N :_{R_S} M_S) \cap R$. Then $IR_S = (N :_{R_S} M_S)$ is a quasi-prime ideal of R_S by assumption. By Remark 2.2, I is a quasi-prime ideal of R so, $N \cap M$ is a quasi-prime submodule of M .
- (6) By assumption, $(N :_R M)$ is a quasi-prime ideal and since R is Laskerian, $(N :_R M)$ is primary. By Remark 2.2 and [Nor68, p. 152, Proposition 8],

$(N_S :_{R_S} M_S) = (N :_R M)R_S$ is a quasi-prime ideal of R_S . So, N_S is a quasi-prime submodule of M_S .

- (7) Let N be a primary submodule of M . Then $(N :_R M)$ is a primary ideal of R , so is quasi-prime by Remark 2.2. Hence, $N \in q\text{Spec}(M)$.
- (8) $IR_{\mathfrak{p}}$ is a proper ideal of $R_{\mathfrak{p}}$. But $R_{\mathfrak{p}}$ is a serial ring. Thus by Remark 2.2, $IR_{\mathfrak{p}}$ is a quasi-prime ideal of $R_{\mathfrak{p}}$ and therefore $S_{\mathfrak{p}}(I) = IR_{\mathfrak{p}} \cap R$ is quasi-prime by Remark 2.2. If R is Laskerian, then by Remark 2.2, $S_{\mathfrak{p}}(I)$ is primary. Let \mathfrak{q} be a prime ideal of R such that $I \subseteq \mathfrak{q} \subseteq \mathfrak{p}$. Then

$$S_{\mathfrak{p}}(I) \subseteq S_{\mathfrak{p}}(\mathfrak{q}) \subseteq S_{\mathfrak{p}}(\mathfrak{p}).$$

By definition, $S_{\mathfrak{p}}(\mathfrak{q}) = \mathfrak{q}$ and $S_{\mathfrak{p}}(\mathfrak{p}) = \mathfrak{p}$. Since $S_{\mathfrak{p}}(I)$ is a \mathfrak{p} -primary ideal of R , we have

$$\mathfrak{p} = \sqrt{S_{\mathfrak{p}}(I)} \subseteq \sqrt{S_{\mathfrak{p}}(\mathfrak{q})} \subseteq \sqrt{S_{\mathfrak{p}}(\mathfrak{p})} = \mathfrak{p}.$$

Therefore, $\mathfrak{q} = \mathfrak{p}$ and \mathfrak{p} is minimal prime ideal over I .

- (9) Since $\mathfrak{p} \in \text{Supp}(M/N)$, $N_{\mathfrak{p}} \neq M_{\mathfrak{p}}$. By assumption $R_{\mathfrak{p}}$ is a serial ring. By part (4), $N_{\mathfrak{p}}$ is a quasi-prime submodule of $M_{\mathfrak{p}}$. By part (5), $S_{\mathfrak{p}}(N) = N_{\mathfrak{p}} \cap M$ is a quasi-prime submodule of M . The last assertion follows from [Lu03, Result 2].
- (10) We have $(N_{\mathfrak{p}} : M_{\mathfrak{p}}) = (N : M)_{\mathfrak{p}} \subseteq \mathfrak{p}R_{\mathfrak{p}}$ and $R_{\mathfrak{p}}$ is a serial ring. So, $N_{\mathfrak{p}}$ is a quasi-prime submodule of $M_{\mathfrak{p}}$.

□

It is shown in [Azi03, Proposition 2.1] that R is a field if every proper submodule of M is a prime submodule of M and $S_{(0)}(\mathbf{0}) \neq M$. In the following, we give an example that shows it is not the case for any quasi-prime submodule.

Example 2.5. (1) Every proper submodule of the \mathbb{Z} -module $M = \mathbb{Z}(p^\infty)$ is a quasi-prime submodule, in which p is a prime integer. For, $(L :_{\mathbb{Z}} M) = (0)$ where L is a submodule of M (see [Lu95, p. 3745]).

(2) Let R be an integral domain which is not a field and K be the field of quotients of R . Then every proper submodule of K is a quasi-prime submodule. Since $xK = K$ for every nonzero element $x \in R$, $(N : K) = (0)$ for every proper submodule N of K .

Theorem 2.6. *Let M be a finitely generated R -module and let I be a primary quasi-prime ideal of R . If S is a multiplicatively closed subset of R such that $I \cap S = \emptyset$, then the map $N \mapsto N_S$ is a surjection from $q\text{Spec}_I(M)$ to $q\text{Spec}_{IR_S}(M_S)$.*

Proof. Let $N \in q\text{Spec}_I(M)$. Since M is finitely generated and $I \cap S = \emptyset$ we have $IR_S = (N :_R M)R_S = (N_S :_{R_S} M_S) \neq R_S$. By Remark 2.2, IR_S is a quasi-prime ideal of R_S . Therefore, N_S is a quasi-prime submodule of M_S . Let L be a quasi-prime submodule of M_S with $(L :_{R_S} M_S) = IR_S$. By Lemma 2.4(5), $L \cap M$ is a quasi-prime submodule of M . Moreover, using that I is primary we have

$$I = IR_S \cap R = (L :_{R_S} M_S) \cap R = ((L \cap M) :_R M).$$

So, $L \cap M$ is a quasi-prime submodule of M .

□

Corollary 2.7. *Let M be a finitely generated R -module and $\mathfrak{p} \in \text{Spec}(R)$.*

- (1) *Let I be a \mathfrak{p} -primary quasi-prime ideal of R . Then the map $N \mapsto N_{\mathfrak{p}}$ is a surjection from $q\text{Spec}_I(M)$ to $q\text{Spec}_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})$.*

- (2) The map $N \mapsto N_{\mathfrak{p}}$ is a surjection from $q\text{Spec}_{\mathfrak{p}}(M)$ to $q\text{Spec}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{Spec}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$.
- (3) Let N be a quasi-prime submodule of M with $(N : M) = \mathfrak{p}$. Then $S_{\mathfrak{p}}(N)$ is a prime submodule minimal over N and any other \mathfrak{p} -prime submodule of M containing N , must contain $S_{\mathfrak{p}}(N)$.

Proof. (1) and (2) follows from Theorem 2.6. For establish (3), note that by part (2), $N_{\mathfrak{p}}$ is a $\mathfrak{p}R_{\mathfrak{p}}$ -prime submodule of $M_{\mathfrak{p}}$ and by [Lu95, Proposition 1], $S_{\mathfrak{p}}(N) = N_{\mathfrak{p}} \cap M$ is a \mathfrak{p} -prime submodule of M . Now the results follows from [Lu03, Result 3]. \square

Definition 2.8. Let M be an R -module. For a submodule N of M we define

$$\begin{aligned} D^M(N) &= \{L \in q\text{Spec}(M) \mid (L : M) \supseteq (N : M)\}, \\ \Omega^M(N) &= \{L \in q\text{Spec}(M) \mid L \supseteq N\}. \end{aligned}$$

If there is no ambiguity we write $D(N)$ (resp. $\Omega(N)$) instead of $D^M(N)$ (resp. $\Omega^M(N)$).

Lemma 2.9. Let M be an R -module with $q\text{Spec}(M) = \emptyset$. Then $\mathfrak{p}M = M$ for every maximal ideal \mathfrak{p} of R . On the other hand, if $IM = M$ for every $I \in D^R(\text{Ann}(M))$, then $q\text{Spec}(M) = \emptyset$.

Definition 2.10. When $q\text{Spec}(M) \neq \emptyset$, the map $\psi : q\text{Spec}(M) \rightarrow q\text{Spec}(R/\text{Ann}(M))$ defined by $\psi(L) = (L : M)/\text{Ann}(M)$ for every $L \in q\text{Spec}(M)$, will be called the natural map of $q\text{Spec}(M)$. An R -module M is called *quasi-primeful* if either $M = (0)$ or $M \neq (0)$ and has a surjective natural map.

Example 2.11. Let $\Sigma := q\text{Spec}(\mathbb{Z}) \setminus \{(0)\}$. Consider the \mathbb{Z} -module $M = \bigoplus_{I \in \Sigma} \mathbb{Z}/I$. We will show that M is a quasi-primeful \mathbb{Z} -module. Note that $(0 : M) = \text{Ann}(M) = (0)$. So, $(0) \in q\text{Spec}_{(0)}(M)$. On the other hand, for each nonzero quasi-prime ideal I of \mathbb{Z} , we have $(IM : M) = I \in q\text{Spec}(\mathbb{Z})$. This implies that $IM \in q\text{Spec}_I(M)$. We conclude that M is a quasi-primeful \mathbb{Z} -module.

Let Y be a subset of $q\text{Spec}(M)$ for an R -module M . We will denote the intersection of all elements in Y by $\mathfrak{I}(Y)$.

Proposition 2.12. Let F be a free R -module and I be a quasi-prime ideal of R . Then

- (1) IF is a quasi-prime submodule, i.e., F is quasi-primeful;
- (2) $IF = \mathfrak{I}(q\text{Spec}_I(F))$;
- (3) If F has primary decomposition for submodules, then I is primary.

Proof. (1) Since F is free we have $I = (IF : F)$, so that IF is a quasi-prime submodule. (2) This is clear by (1). For (3), Let $\cap_{i=1}^n Q_i$ be a primary decomposition of IF , where each Q_i is a \mathfrak{p}_i -primary submodule of F . Then $I = (IF : F) = \cap_{i=1}^n (Q_i : F)$. Since I is quasi-prime, $I = (Q_j : F)$ for some $1 \leq j \leq n$. Hence, I is primary since Q_j is a primary submodule. \square

Lemma 2.13. Let M, M_1, M_2 be R -modules such that $M = M_1 \oplus M_2$ and $I \in D^R(\text{Ann}(M))$. If $N \in q\text{Spec}_I(M_1)$ (resp. $N \in q\text{Spec}_I(M_2)$), then $N \oplus M_2 \in q\text{Spec}_I(M)$ (resp. $M_1 \oplus N \in q\text{Spec}_I(M)$). In particular, every direct sum of a finite number of quasi-primeful R -modules is quasi-primeful over R .

Proof. This is straightforward and we omit it. \square

Theorem 2.14. *Let M be an R -module. Then M is quasi-primeful in each of the following cases:*

- (1) R is PID and M is finitely generated;
- (2) R is Dedekind domain and M is faithfully flat;
- (3) R is Laskerian and M is locally free.

Proof. (1) Let N be a cyclic submodule of M and $I \in D^R(\text{Ann}(N))$. Then $N = R/\text{Ann}(m)$ for some $m \in N$ and $(I/\text{Ann}(N) : N) = I$. Hence, N is quasi-primeful. It is well-known that a finitely generated module over a PID is finite direct sum of cyclic submodules. Hence, in the light of Lemma 2.13, M is quasi-primeful. (2) Let $J \in q\text{Spec}(R)$. Since M is faithfully flat, $JM \neq M$ and by Remark 2.2, J is primary. So, JM is a primary submodule by [Lu84, Theorem 3], and $(JM : M) = J$ is a quasi-prime ideal of R , i.e., JM is quasi-prime. (3) Let $I \in D(\text{Ann}(M))$. Since R is Laskerian, $\mathfrak{p} := \sqrt{I}$ is a prime ideal of R and $IR_{\mathfrak{p}}$ is a quasi-prime ideal of $R_{\mathfrak{p}}$ by Remark 2.2(6). Since $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module, there exists a quasi-prime submodule N of $M_{\mathfrak{p}}$ such that $(N :_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) = IR_{\mathfrak{p}}$ by Proposition 2.12. Now, $(N \cap M : M) = IR_{\mathfrak{p}} \cap R = I$ by Lemma 2.4. This implies that M is quasi-primeful. \square

We note that not every quasi-primeful module is finitely generated. For example, every (finite or infinite dimensional) vector space is quasi-primeful.

Remark 2.15. (See [EBS88, Theorem 3.1]) Let M be a faithful multiplication module over R . Then M is finitely generated if and only if $\mathfrak{m}M \neq M$ for every maximal ideal \mathfrak{m} of R .

Proposition 2.16. *Let M be a nonzero quasi-primeful R -module.*

- (1) *Let I be a radical ideal of R . Then $(IM : M) = I$ if and only if $\text{Ann}(M) \subseteq I$;*
- (2) *$\mathfrak{p}M \in q\text{Spec}(M)$ for every $\mathfrak{p} \in V(\text{Ann}(M))$;*
- (3) *$\mathfrak{p}M \in \text{Spec}_{\mathfrak{p}}(M)$ for every $\mathfrak{p} \in V(\text{Ann}(M)) \cap \text{Max}(R)$;*
- (4) *If $\dim(R) = 0$, then M is primeful;*
- (5) *If M is multiplication, then M is finitely generated.*

Proof. (1) The necessity is clear. For sufficiency, we note that $\text{Ann}(M) \subseteq I = \cap_i \mathfrak{p}_i$, where \mathfrak{p}_i runs through $V^R(I)$ since I is a radical ideal. On the other hand, M is quasi-primeful and $\mathfrak{p}_i \in D(\text{Ann}(M))$ so, there exists a quasi-prime submodule L_i such that $(L_i : M) = \mathfrak{p}_i$. Now, we obtain that

$$I \subseteq (IM : M) = ((\cap_i \mathfrak{p}_i)M : M) \subseteq \cap_i (\mathfrak{p}_i M : M) \subseteq \cap_i (L_i : M) = \cap_i \mathfrak{p}_i = I.$$

Thus $(IM : M) = I$. (2) and (3) follows from part (1). For (4), let $\mathfrak{p} \in V^R(\text{Ann}(M))$. Then by part (3), $\mathfrak{p}M \neq M$ and by [Lu07, Result 3], M is primeful. (5) Since M is a faithful multiplication module over $R/\text{Ann}(M) = \bar{R}$ and $\bar{\mathfrak{m}}M \neq M$ for every $\bar{\mathfrak{m}} \in \text{Max}(\bar{R})$ by (3), M is finitely generated over \bar{R} by Remark 2.15. Hence, M is finitely generated over R . \square

Corollary 2.17. *Let M be an R -module.*

- (1) *Let M be a quasi-primeful R -module. If I is an ideal of R contained in the Jacobson radical $\text{Rad}(R)$ such that $IM = M$, then $M = (0)$.*

- (2) Let R be a PID and M be torsion-free. Then M is quasi-primeful if $pM \neq M$ for every irreducible element $p \in R$.
- (3) If M is faithful quasi-primeful, then M is flat if and only if M is faithfully flat.
- (4) If M is projective and R is Laskerian, then M is quasi-primeful.

Proof. (1) Suppose that $M \neq (0)$. Then $\text{Ann}(M) \neq R$. If \mathfrak{m} is any maximal ideal containing $\text{Ann}(M)$, then $I \subseteq \text{Rad}(R) \subseteq \mathfrak{m}$ and $IM = M = \mathfrak{m}M$ whence $(\mathfrak{m}M : M) = R \neq \mathfrak{m}$, a contradiction to Proposition 2.16. (2) If for every irreducible element $p \in R$, $pM \neq M$, then M is faithfully flat and by Theorem 2.14, M is quasi-primeful. (3) The sufficiency is clear. Suppose that M is flat. By Proposition 2.16, for every $\mathfrak{p} \in \text{Max}(R) \subseteq D(0)$, $\mathfrak{p}M \neq M$. This implies that M is faithfully flat. (4) Since every projective module is locally free, by Theorem 2.14, M is quasi-primeful. \square

Example 2.18. The \mathbb{Z} -module \mathbb{Q} is a flat and faithful, but not faithfully flat. So, \mathbb{Q} is not quasi-primeful.

We give an elementary example of a module which is not quasi-primeful. If R is a domain, then an R -module M is *divisible* if $M = rM$ for all nonzero elements $r \in R$. We note that every injective module is divisible.

Proposition 2.19. *Let R be a domain which is not a field. Then every nonzero divisible R -module is not quasi-primeful.*

Proof. By assumption $\text{Ann}(M) = (0)$ and there exists a nonzero prime ideal \mathfrak{p} of R . Hence $\mathfrak{p} \in V^R(\text{Ann}(M))$ and $\mathfrak{p}M = M$. Therefore, M is not quasi-primeful by Proposition 2.16. \square

Proposition 2.20. *Let R be a domain over which every module is quasi-primeful. Then R is a field.*

Proof. Suppose that R is not a field. Then its field K of quotients is a nonzero divisible R -module. Hence, K is not quasi-primeful over R by Proposition 2.19, which is a contradiction to the definition of R . \square

An R -module M is called *weak multiplication* if $\text{Spec}(M) = \emptyset$ or for every prime submodule N of M , we have $N = IM$, where I is an ideal of R . One can easily show that if M is a weak multiplication module, then $N = (N : M)M$ for every prime submodule N of M ([AS95] and [Azi03]). As is seen in [AS95], \mathbb{Q} is a weak multiplication \mathbb{Z} -module which is not a multiplication module.

Definition 2.21. An R -module M is called *quasi-prime-embedding*, if the natural map $\psi : q\text{Spec}(M) \rightarrow q\text{Spec}(R/\text{Ann}(M))$ is injective.

We will show that every cyclic module is quasi-prime-embedding (Corollary 2.23). Thus any ring R as R -module is quasi-prime-embedding.

Proposition 2.22. *The following statements are equivalent for any R -module M :*

- (1) M is quasi-prime-embedding;
- (2) If $D(L) = D(N)$, then $L = N$, for any $L, N \in q\text{Spec}(M)$;
- (3) $|q\text{Spec}_I(M)| \leq 1$ for every $I \in q\text{Spec}(R)$.

Proof. (1) \Rightarrow (2) Let $D(L) = D(N)$. Then $(L : M) = (N : M)$. Now by (1), $L = N$. (2) \Rightarrow (3) Suppose that $L, N \in q\text{Spec}_I(M)$ for some $I \in q\text{Spec}(R)$. Hence $(L : M) = (N : M) = I$ and so, $D(L) = D(N)$. Thus, $L = N$ by (2). (3) \Rightarrow (1) Let $\bar{I} := \psi(L) = \psi(N)$. Then $I = (L : M) = (N : M)$. By (3), $L = N$, and so ψ is injective. \square

Corollary 2.23. *Consider the following statements for an R -module M :*

- (1) M is multiplication;
- (2) M is quasi-prime-embedding;
- (3) M is weak multiplication;
- (4) $|\text{Spec}_{\mathfrak{p}}(M)| \leq 1$ for every prime ideal \mathfrak{p} of R ;
- (5) M/pM is cyclic for every maximal ideal \mathfrak{p} of R .

Then (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5). Further, if M is finitely generated, then (5) \Rightarrow (1).

Proof. (1) \Rightarrow (2) Let $D(N) = D(L)$ for $N, L \in q\text{Spec}(M)$. Then $(N : M) = (L : M)$ and since M is multiplication, $N = L$. Therefore, (2) follows from Proposition 2.22. (2) \Rightarrow (3) Let P be a \mathfrak{p} -prime submodule of M . By Lemma 2.4, $(P : M)M \in q\text{Spec}_{\mathfrak{p}}(M)$. Combining this fact with Proposition 2.22, we obtain that $P = (P : M)M$. This yields M is weak multiplication. (3) \Rightarrow (4) The case $\text{Spec}_{\mathfrak{p}}(M) = \emptyset$ is trivially true. Let $P, Q \in \text{Spec}_{\mathfrak{p}}(M)$ for some prime ideal \mathfrak{p} of R . Then $(P : M) = (Q : M)$. Therefore $P = (P : M)M = (Q : M)M = Q$. The (4) \Rightarrow (5) and last statement is true due to [MMS97, Theorem 3.5]. \square

An R -module M is called *locally cyclic* if $M_{\mathfrak{p}}$ is a cyclic module over the local ring $R_{\mathfrak{p}}$ for every prime ideal \mathfrak{p} of R . Multiplication modules are locally cyclic (see [EBS88, Theorem 2.2]).

Theorem 2.24. *Let M be an R -module and let S be a multiplicatively closed subset of R .*

- (1) *If M is Laskerian quasi-prime-embedding, then every quasi-prime submodule of M is primary (see [Azi08, Theorem 2.1]).*
- (2) *Let R be a serial ring. Then M is multiplication if and only if M is quasi-prime-embedding.*
- (3) *If M is quasi-prime-embedding, then $S^{-1}M$ is also a quasi-prime-embedding $S^{-1}R$ -module.*
- (4) *If M is free, Then M is quasi-prime-embedding if and only if M is cyclic.*
- (5) *If M is projective quasi-prime-embedding, then M is locally cyclic.*
- (6) *If R is an arithmetical ring and M is quasi-prime-embedding, then M is locally cyclic.*
- (7) *Let R be a semi-local arithmetical ring. Then M is cyclic if and only if M is quasi-prime-embedding.*
- (8) *A finitely generated module M is locally cyclic if and only if M is multiplication if and only if M is quasi-prime-embedding.*
- (9) *Let R be a Dedekind domain and M be a non-faithful quasi-prime-embedding R -module. Then M is cyclic.*

Proof. (1) Let P be a quasi-prime submodule of M and $\bigcap_{i=1}^m N_i$ be a primary decomposition for P . Since P is quasi-prime,

$$(N_j : M) \subseteq (P : M) = \bigcap_{i=1}^m (N_i : M) \subseteq (N_j : M)$$

for some $1 \leq j \leq m$. Hence, N_j is a quasi-prime submodule and by Proposition 2.22, $P = N_j$.

(2) The necessity follows from Corollary 2.23. Let N be a proper submodule of M . By Lemma 2.4, N and $(N : M)M$ are quasi-prime submodules of M . Therefore, $N = (N : M)M$ by Proposition 2.22, and so M is multiplication.

(3) Use Lemma 2.4 and Proposition 2.22.

(4) If M is cyclic, then M is quasi-prime-embedding by Corollary 2.23. We assume that M is quasi-prime-embedding and M is not cyclic. Hence, $M = \bigoplus_{i \in I} R$, where $|I| > 1$. Let $\mathfrak{p} \in \text{Spec}(R)$ and α, β be two distinct elements of I . It is easy to see that

$$N = \mathfrak{p} \oplus \left(\bigoplus_{\substack{i \in I \\ i \neq \alpha}} R \right) \quad \text{and} \quad L = \mathfrak{p} \oplus \left(\bigoplus_{\substack{i \in I \\ i \neq \beta}} R \right)$$

are two distinct quasi-prime submodules of M with $(N : M) = (L : M) = \mathfrak{p}$. By Proposition 2.22, $N = L$, a contradiction.

(5) Let $\mathfrak{p} \in \text{Spec}(R)$. Then by (3), $M_{\mathfrak{p}}$ is quasi-prime-embedding. On the other hand, $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module. Hence, $M_{\mathfrak{p}}$ is a cyclic $R_{\mathfrak{p}}$ -module by (4).

(6) For each $\mathfrak{p} \in \text{Spec}(R)$, $R_{\mathfrak{p}}$ is a serial ring by [Jen66, Theorem 1], and $M_{\mathfrak{p}}$ is quasi-prime-embedding by (3). By (2), $M_{\mathfrak{p}}$ is a multiplication $R_{\mathfrak{p}}$ -module. Therefore, $M_{\mathfrak{p}}$ is cyclic, since $R_{\mathfrak{p}}$ is a quasi-local ring.

(7) Let $\mathfrak{m}_1, \dots, \mathfrak{m}_t$ be all maximal ideals of R . By (6), $M_{\mathfrak{m}_i}$ is a cyclic $R_{\mathfrak{m}_i}$ -module for each i . Hence, M is cyclic by [Bar81, Lemma 3]. Other side is true by Corollary 2.23.

(8) Use [Bar81, Proposition 5] and Corollary 2.23.

(9) By assumption there exist only finitely many prime (maximal) ideal containing $\text{Ann}(M)$. So, by (6), and [Bar81, Lemma 3], M is cyclic. \square

A submodule S of an R -module M will be called *semiprime* if S is an intersection of prime submodules. A prime submodule K of M is said to be *extraordinary* if whenever N and L are semiprime submodules of M with $N \cap L \subseteq K$, then $N \subseteq K$ or $L \subseteq K$. An R -module M is said to be a *top module* if every prime submodule of M is extraordinary. Every multiplication or locally cyclic module is a top module (see [MMS97]). Corollary 2.23 and Theorem 2.24 are very interesting for us, because there is a close relationship between those and top modules. We find the relations between parts (1)-(4) of Corollary 2.23 and top modules. By [MMS97, Theorem 3.5], every multiplication module is top. So we consider part (2) of Corollary 2.23. By Theorem 2.24, every projective quasi-prime-embedding module and every quasi-prime-embedding module over arithmetical ring is locally cyclic, so is top due to [MMS97, Theorem 4.1]. In the next theorem we will show the relationship between part (3) and part (4) of Corollary 2.23 and top modules.

Theorem 2.25. *Let R be a one dimensional Noetherian domain and let M be a nonzero R -module. Then M is a top module in each of the following cases:*

- (1) M is weak multiplication.
- (2) For every prime ideal $\mathfrak{p} \in \text{Spec}(R)$, $|\text{Spec}_{\mathfrak{p}}(M)| \leq 1$ and $S_{(0)}(\mathbf{0}) \subseteq \text{rad}(\mathbf{0})$.

Proof. (1) Let P be a \mathfrak{p} -prime submodule of M and let N and L be non-zero semiprime submodules of M such that $N \cap L \subseteq P$. It is enough to show that $N \subseteq P$ or $L \subseteq P$. If $(N : M)$ or $(L : M) \not\subseteq (P : M)$, then $L \subseteq P$ or $N \subseteq P$ by [Lu89, Lemma 2]. Hence, we consider just the case that $(L : M) \subseteq (P : M)$ and $(N : M) \subseteq (P : M)$. Now, we are going to show that if $N \not\subseteq P$, then $L \subseteq P$. For that, choose $x \in N \setminus P$. So, $x \notin L$. If $(L : x) = (0)$, then $x + L \not\subseteq S_{(0)}(\mathbf{0}_{M/L})$, so $S_{(0)}(\mathbf{0}_{M/L}) \neq M/L$. Since M is weak multiplication, it follows that M/L is also a weak multiplication module. But every weak multiplication module over an integral domain is either torsion or torsion-free (see [Azi03, Proposition 3]). Hence M/L is a torsion-free R -module.

On the other hand, we have $(L : M) \subseteq (L : x) = (0)$. Thus $L \in \text{Spec}_{(0)} M$ by [Lu84, Theorem 1]. Therefore $L = (0)M = (\mathbf{0}) \subseteq P$ as desired. Now let $(L : x) \neq (0)$ and $L = \bigcap_{\lambda \in \Lambda} P_\lambda$, where P_λ are \mathfrak{p}_λ -prime submodules of M for each $\lambda \in \Lambda$. By assumption $P_\lambda = \mathfrak{p}_\lambda M$. This implies that

$$(L : x) = \left(\bigcap_{\lambda \in \Lambda} \mathfrak{p}_\lambda M : x \right) = \bigcap_{\lambda \in \Lambda} (\mathfrak{p}_\lambda M : x).$$

Suppose that Λ' be a subset of Λ such that for each $\lambda \in \Lambda'$, $x \notin \mathfrak{p}_\lambda M$. Since $x \notin L$, hence $\Lambda' \neq \emptyset$. Now by [MS02, Lemma 2.12] and since $\dim(R) = 1$,

$$(0) \neq (L : x) = \bigcap_{\lambda \in \Lambda'} (\mathfrak{p}_\lambda M : x) = \bigcap_{\lambda \in \Lambda'} \mathfrak{p}_\lambda \subseteq (P : M).$$

Therefore, $(L : x)$ is a nonzero ideal of R , and so it is contained in only finitely many prime ideal by [AM69, Proposition 9.1]. Thus, Λ' is a finite set. It follows that there exists $\mathfrak{q} \in \Lambda'$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. This yields $L \subseteq \mathfrak{p}M = P$ as desired.

- (2) If $S_{(0)}(\mathbf{0}) = M$, then $\text{rad}(\mathbf{0}) = M$, i.e., $\text{Spec}(M) = \emptyset$, and so we are done. Therefore, we assume that $S_{(0)}(\mathbf{0}) \neq M$. In this case $S_{(0)}(\mathbf{0})$ is a (0) -prime submodule of M by [Lu03, Lemma 4.5]. We are going to show that every prime submodule of M is extraordinary. Let P be a prime submodule of M and let N and L be two nonzero semiprime submodules of M such that $N \cap L \subseteq P$. In view of above arguments we take $x \in N \setminus P$. If $(L : x) = (0)$, then $(L : M) = (0)$ and by [Lu03, Result 1],

$$S_{(0)}(\mathbf{0}_{M/L}) = S_{(0)}(\mathbf{0})/L \subseteq \text{rad}(\mathbf{0})/L = (\mathbf{0}).$$

Therefore, M/L is a torsion-free R -module and L is a (0) -prime submodule of M by [Lu84, Theorem 1]. By assumption of this part, $L = S_{(0)}(\mathbf{0}) \subseteq \text{rad}(\mathbf{0}) \subseteq P$. Let $(L : x) \neq (0)$ and let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a collection of \mathfrak{p}_λ -prime submodules of M such that $L = \bigcap_{\lambda \in \Lambda} P_\lambda$. If $\mathfrak{p}_k = (0)$ for some $k \in \Lambda$, then $(P_k : M) = (S_{(0)}(\mathbf{0}) : M) = (0)$. Hence, $L \subseteq P_k = S_{(0)}(\mathbf{0}) \subseteq \text{rad}(\mathbf{0}) \subseteq P$. Therefore, we may assume that $\mathfrak{p}_\lambda \neq (0)$ for each $\lambda \in \Lambda$. Since $\dim(R) = 1$, we have $\mathfrak{p}_\lambda = (\mathfrak{p}_\lambda M : M) = (P_\lambda : M)$. Therefore, $\mathfrak{p}_\lambda M$ is a \mathfrak{p}_λ -prime submodule of M by [Lu84, Proposition 2]. By assumption of this part, $P_\lambda = \mathfrak{p}_\lambda M$. This implies that

$$(L : x) = \left(\bigcap_{\lambda \in \Lambda} \mathfrak{p}_\lambda M : x \right) = \bigcap_{\lambda \in \Lambda} (\mathfrak{p}_\lambda M : x).$$

Suppose that Λ' be a subset of Λ such that for each $\lambda \in \Lambda'$, $x \notin \mathfrak{p}_\lambda M$. Since $x \notin L$, hence $\Lambda' \neq \emptyset$. Now, from the [MS02, Lemma 2.12], we have,

$$(0) \neq (L : x) = \bigcap_{\lambda \in \Lambda'} (\mathfrak{p}_\lambda M : x) = \bigcap_{\lambda \in \Lambda'} \mathfrak{p}_\lambda \subseteq (P : M).$$

By [AM69, Proposition 9.1], $(L : x)$ is contained in finitely many prime ideal, i.e., Λ' is finite. So, there exists some $\lambda \in \Lambda'$ such that $\mathfrak{p}_\lambda \subseteq (P : M)$. Therefore, $L \subseteq P$. □

The next example shows that Part (1) of Theorem 2.25 is different from Part (2).

Example 2.26. Consider the \mathbb{Z} -module $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}$. It is easy to see that for every prime ideal $\mathfrak{p} \in \text{Spec}(\mathbb{Z})$, $|\text{Spec}_{\mathfrak{p}}(M)| \leq 1$ and $S_{(0)}(\mathbf{0}) = \text{rad}(\mathbf{0})$. By Theorem 2.25, M is a top module. We note that M is not weak multiplication.

3. SOME TOPOLOGICAL PROPERTIES OF $q\text{Spec}(M)$

Let M be an R -module. Then for submodules N , L and N_i of M we have

- (1) $D(\mathbf{0}) = q\text{Spec}(M)$ and $D(M) = \emptyset$,
- (2) $\bigcap_{i \in I} D(N_i) = D(\sum_{i \in I} (N_i : M)M)$,
- (3) $D(N) \cup D(L) = D(N \cap L)$.

Now, we put

$$\zeta(M) = \{ D(N) \mid N \leq M \}.$$

From (1), (2) and (3) above, it is evident that for any module M there exists a topology, τ say, on $q\text{Spec}(M)$ having $\zeta(M)$ as the family of all closed sets. The topology τ is called the *developed Zariski topology on $q\text{Spec}(M)$* . For the reminder of this paper, for every ideal $I \in D(\text{Ann}(M))$, \overline{R} and \overline{I} will denote respectively $R/\text{Ann}(M)$ and $I/\text{Ann}(M)$. Let Y be a subset of $q\text{Spec}(M)$ for an R -module M . We will denote the intersection of all elements in Y by $\mathfrak{S}(Y)$ and the closure of Y in $q\text{Spec}(M)$ with respect to the developed Zariski topology by $Cl(Y)$. The proof of next lemma is easy.

Lemma 3.1. *Let I be a proper ideal of R and M be an R -module with submodules N and L . Then we have*

- (1) *If $(N : M) = (L : M)$, then $D(N) = D(L)$. The converse is also true if both N and L are quasi-prime submodules of M ;*
- (2) $D(N) = \bigcup_{I \in D^{R(N:M)} q\text{Spec}_I(M)$;
- (3) $D(N) = D((N : M)M) = \Omega^M((N : M)M)$;
- (4) *Let Y be a subset of $q\text{Spec}(M)$. Then $Y \subseteq D(N)$ if and only if $(N : M) \subseteq (\mathfrak{S}(Y) : M)$.*

Proposition 3.2. *Let M be an R -module and $\psi : q\text{Spec}(M) \rightarrow q\text{Spec}(R/\text{Ann}(M))$ be the natural map.*

- (1) *The natural map ψ is continuous with respect to the developed Zariski topology.*
- (2) *If M is quasi-primeful, then ψ is both closed and open; more precisely, for every submodule N of M , $\psi(D^M(N)) = D^{\overline{R}}(\overline{(N : M)})$ and $\psi(q\text{Spec}(M) - D^M(N)) = q\text{Spec}(\overline{R}) - D^{\overline{R}}(\overline{(N : M)})$.*
- (3) *ψ is bijective if and only if it is a homeomorphism.*

Proof. (1) Let I be an ideal of R containing $\text{Ann}(M)$ and let $L \in \psi^{-1}(D^{\bar{R}}(\bar{I}))$. There exists some $\bar{J} \in D^{\bar{R}}(\bar{I})$ such that $\psi(L) = \bar{J}$. Hence, $J = (L : M) \supseteq I$ and $L \in D^M(IM)$. Now, let $K \in D^M(IM)$. Then $(K : M) \supseteq (IM : M) \supseteq I$, and so $K \in \psi^{-1}(D^{\bar{R}}(\bar{I}))$. Consequently, $\psi^{-1}(D^{\bar{R}}(\bar{I})) = D^M(IM)$, i.e., ψ is continuous. (2) By part (1), ψ is a continuous map such that $\psi^{-1}(D^{\bar{R}}(\bar{I})) = D^M(IM)$ for every ideal I of R containing $\text{Ann}(M)$. Hence, for every submodule N of M , $\psi^{-1}(D^{\bar{R}}(\overline{(N : M)})) = D^M((N : M)M) = D^M(N)$. Since the natural map ψ is surjective, $\psi(D^M(N)) = \psi \circ \psi^{-1}(D^{\bar{R}}(\overline{(N : M)})) = D^{\bar{R}}(\overline{(N : M)})$. Similarly, $\psi(q\text{Spec}(M) - D^M(N)) = q\text{Spec}(\bar{R}) - D^{\bar{R}}(\overline{(N : M)})$. (3) This follows from (1) and (2). \square

Theorem 3.3. *Let M be a quasi-primeful R -module. Then the following statements are equivalent:*

- (1) $q\text{Spec}(M)$ is connected;
- (2) $q\text{Spec}(\bar{R})$ is connected;
- (3) The ring \bar{R} contains no idempotent other than $\bar{0}$ and $\bar{1}$.

Consequently, if R is a quasi-local ring, then both $q\text{Spec}(M)$ and $q\text{Spec}(\bar{R})$ are connected.

Proof. (1) \Rightarrow (2) follows from that ψ is a surjective and continuous.

For (2) \Rightarrow (1), we assume that $q\text{Spec}(\bar{R})$ is connected. If $q\text{Spec}(M)$ is disconnected, then $q\text{Spec}(M)$ must contain a non-empty proper subset Y that is both open and closed. Accordingly, $\psi(Y)$ is a non-empty subset of $q\text{Spec}(\bar{R})$ that is both open and closed by Proposition 3.2. To complete the proof, it suffices to show that $\psi(Y)$ is a proper subset of $q\text{Spec}(\bar{R})$ so that $q\text{Spec}(\bar{R})$ will be disconnected.

Since Y is open, $Y = q\text{Spec}(M) - D^M(N)$ for some $N \leq M$ whence $\psi(Y) = q\text{Spec}(\bar{R}) - D^{\bar{R}}(\overline{(N : M)})$ by Proposition 3.2 again. Therefore, if $\psi(Y) = q\text{Spec}(\bar{R})$, then $D^{\bar{R}}(\overline{(N : M)}) = \emptyset$, and so $\overline{(N : M)} = \bar{R}$, i.e., $N = M$. It follows that $Y = q\text{Spec}(M) - D^M(N) = q\text{Spec}(M) - D^M(M) = q\text{Spec}(M)$ which is impossible. Thus $\psi(Y)$ is a proper subset of $q\text{Spec}(\bar{R})$.

For (2) \Leftrightarrow (3), it is enough for us to show that $q\text{Spec}(R)$ is disconnected if and only if R has an idempotent $e \neq 0, 1$. Suppose that $e \neq 0, 1$ is an idempotent in R . Hence $R = Re \oplus R(1 - e)$. It follows that $q\text{Spec}(R) = (q\text{Spec}(R) \setminus D^R(Re)) \cup (q\text{Spec}(R) \setminus D^R(R(1 - e)))$ and $\emptyset = (q\text{Spec}(R) \setminus D^R(Re)) \cap (q\text{Spec}(R) \setminus D^R(R(1 - e)))$. This implies that $q\text{Spec}(R)$ is disconnected. Now, we assume that $q\text{Spec}(R)$ is disconnected. Thus $q\text{Spec}(R) = D^R(I) \cup D^R(J)$ where I and J are two ideals of R . We have that $q\text{Spec}(R) = D^R(I \cap J)$ and so, $I \cap J \subseteq \mathfrak{S}(q\text{Spec}(R))$. Also, $\emptyset = D^R(I) \cap D^R(J) = D^R(I + J)$. This implies that $I + J = R$. There exist $a \in I$ and $b \in J$ such that $a + b = 1$. On the other hand,

$$ab \in IJ \subseteq I \cap J \subseteq \mathfrak{S}(q\text{Spec}(R)) \subseteq \sqrt{(0)}.$$

So, $(ab)^n = 0$ for some $n \in \mathbb{N}$. We have $1 = (a + b)^n = a^n + b^n + abx$ where $x \in R$. Since $abx \in \sqrt{(0)} \subseteq \text{Rad}(R)$, $a^n + b^n$ is a unit in R . Let u be the inverse of $a^n + b^n$. Note that $ua^n b^n = 0$. Thus

$$ua^n = ua^n(u(a^n + b^n)) = u^2 a^{2n} + u^2 a^n b^n = (ua^n)^2.$$

Similarly, $ub^n = (ub^n)^2$. If $ua^n = 0$, then $a^n = 0$, and so $1 = b(b^{n-1} + ax) \in J$ which is contradiction because $D^R(J) \neq \emptyset$. Consequently, ua^n and ub^n are nonzero.

On the other hand, if $ua^n = ub^n = 1$, then $1 = u(a^n + b^n) = ua^n + ub^n = 1 + 1$, which is contradiction. We conclude that either ua^n or ub^n is idempotent. \square

Proposition 3.4. *Let M be an R -module, $Y \subseteq q\text{Spec}(M)$ and let $L \in q\text{Spec}_I(M)$.*

- (1) $D(\mathfrak{S}(Y)) = Cl(Y)$. In particular $Cl(\{L\}) = D(L)$;
- (2) Let M be a semiprimitive (resp. reduced) R -module and $\text{Max}(M)$ (resp. $\text{Spec}(M)$) be a non-empty connected subspace of $q\text{Spec}(M)$. Then $q\text{Spec}(M)$ is connected;
- (3) If $(\mathbf{0}) \in Y$, Then Y is dense in $q\text{Spec}(M)$.
- (4) The set $\{L\}$ is closed in $q\text{Spec}(M)$ if and only if
 - (a) I is a maximal element in $\{(N : M) \mid N \in q\text{Spec}(M)\}$, and
 - (b) $q\text{Spec}_I(M) = \{L\}$.
- (5) If $\{L\}$ is closed in $q\text{Spec}(M)$, then L is a maximal element of $q\text{Spec}(M)$ and $q\text{Spec}_I(M) = \{L\}$.
- (6) M is quasi-prime-embedding if and only if $q\text{Spec}(M)$ is a T_0 -space.
- (7) $q\text{Spec}(M)$ is a T_1 -space if and only if $q\text{Spec}(M)$ is a T_0 -space and for every element $L \in q\text{Spec}(M)$, $(L : M)$ is a maximal element in $\{(N : M) \mid N \in q\text{Spec}(M)\}$.
- (8) If $q\text{Spec}(M)$ is a T_1 -space, then $q\text{Spec}(M)$ is a T_0 -space and every quasi-prime submodule is a maximal element of $q\text{Spec}(M)$. The converse is also true, when M is finitely generated.
- (9) Let $(\mathbf{0}) \in q\text{Spec}(M)$. Then $q\text{Spec}(M)$ is a T_1 -space if and only if $(\mathbf{0})$ is the only quasi-prime submodule of M .

Proof. (1) Clearly, $Y \subseteq D(\mathfrak{S}(Y))$. Next, let $D(N)$ be any closed subset of $q\text{Spec}(M)$ containing Y . Then $(L : M) \supseteq (N : M)$ for every $L \in Y$ so that $(\mathfrak{S}(Y) : M) \supseteq (N : M)$. Hence, for every $Q \in D(\mathfrak{S}(Y))$, $(Q : M) \supseteq (\mathfrak{S}(Y) : M) \supseteq (N : M)$, namely $D(\mathfrak{S}(Y)) \subseteq D(N)$. This proves that $D(\mathfrak{S}(Y))$ is the smallest closed subset of $q\text{Spec}(M)$ containing Y , hence $D(\mathfrak{S}(Y)) = Cl(Y)$.

- (2) Let M be reduced. Then by (1), we have $Cl(\text{Spec}(M)) = D(\mathfrak{S}(\text{Spec}(M))) = D(\mathbf{0}) = q\text{Spec}(M)$. Therefore, $q\text{Spec}(M)$ is connected by [Mun99, p.150, Theorem 23.4]. A similar proof is true for semiprimitive modules.
- (3) is clear by (1).
- (4) Suppose that $\{L\}$ is closed. Then $\{L\} = D(L)$ by (1). Let $N \in q\text{Spec}(M)$ such that $(L : M) \subseteq (N : M)$. Hence, $N \in D(L) = \{L\}$, and so $q\text{Spec}_I(M) = \{L\}$, where $I = (L : M)$. On the other hand we assume that (a) and (b) hold. Let $N \in Cl(\{L\})$. Hence, $(N : M) \supseteq (L : M)$ by (1). By (a), $(N : M) = (L : M)$. So, $L = N$ by (b). This yields $Cl(\{L\}) = \{L\}$.
- (5) Let $P \in q\text{Spec}(M)$ such that $L \subseteq P$. Then $(L : M) \subseteq (P : M)$. i.e., $P \in D(L) = Cl(\{L\}) = \{L\}$. Hence, $P = L$, and so L is a maximal element of $q\text{Spec}(M)$.
- (6) We recall that a topological space is T_0 if and only if the closures of distinct points are distinct. Now, the result follows from part (1) and Proposition 2.22.
- (7) We recall that a topological space is T_1 if and only if every singleton subset is closed. The result follows from (4), (5) and (6).

- (8) Trivially, $q\text{Spec}(M)$ is a T_0 -space and every its singleton subset is closed. Every quasi-prime submodule is a maximal element of $q\text{Spec}(M)$ by (5). Now, we suppose that M is finitely generated. Thus, every quasi-prime submodule is maximal. Let $N \in q\text{Spec}(M)$ such that $N \in Cl(\{L\}) = D(L)$. Since L is maximal, $(L : M) = (N : M)$. By Proposition 2.22, $N = L$. Hence, every singleton subset of $q\text{Spec}(M)$ is closed. So, $q\text{Spec}(M)$ is a T_1 -space.
- (9) Use part (8). □

Example 3.5. Consider the \mathbb{Z} -module $M = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$, where p runs through the set of all prime integers. We will show that $q\text{Spec}(M)$ is not a T_1 -space. Note that $(0 : M) = \text{Ann}(M) = (0)$. Hence, $(0) \in q\text{Spec}(M)$. On the other hand, for each quasi-prime ideal I of \mathbb{Z} , we have $(IM : M) = \sqrt{I} \in q\text{Spec}(\mathbb{Z})$. So, $q\text{Spec}(M)$ is infinite and $q\text{Spec}(M)$ is not a T_1 -space by Proposition 3.4.

Remark 3.6. Let M be a finitely generated (or co-semisimple) R -module. Since every quasi-prime submodule is contained in a maximal submodule, $q\text{Spec}(M)$ is a T_1 -space if and only if $q\text{Spec}(M)$ is a T_0 -space and $q\text{Spec}(M) = \text{Max}(M)$. Since $q\text{Spec}(R)$ is always a T_0 -space (see [Azi08, Theorem 4.1]), we have $q\text{Spec}(R)$ is a T_1 -space if and only if $q\text{Spec}(R) = \text{Max}(R)$. If R is absolutely flat, then by [Azi08, Theorem 2.1], $q\text{Spec}(R) = \text{Spec}(R) = \text{Max}(R)$. Therefore, $q\text{Spec}(R)$ is a T_1 -space. It is clear that if M is free, then $q\text{Spec}(M)$ is a T_1 -space if and only if M is isomorphic to R and $q\text{Spec}(R)$ is a T_1 -space.

Theorem 3.7. *Let M be a finitely generated R -module. The following statements are equivalent*

- (1) $q\text{Spec}(M)$ is a T_1 -space.
- (2) M is a multiplication module and $q\text{Spec}(M) = \text{Max}(M)$.

Proof. Use Corollary 2.23, Remark 3.6 and Proposition 3.4(6). □

Corollary 3.8. *Let M be an R -module.*

- (1) *Let R be an integral domain. If $q\text{Spec}(R)$ is a T_1 -space, then R is a field.*
- (2) *If M is Noetherian and $q\text{Spec}(M)$ is a T_1 -space, then M is Artinian cyclic.*

Proof. (1) By Remark 3.6, we have $q\text{Spec}(R) = \text{Max}(R)$. But $(0) \in q\text{Spec}(R)$ by assumption. Hence, R is a field. (2) By Theorem 3.7, M is multiplication and every prime submodule of M is maximal. By [Beh06, Theorem 4.9], M is Artinian. The result follows from [EBS88, Corollary 2.9]. □

A topological space X is said to be *irreducible* if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X . A topological space X is irreducible if for any decomposition $X = A_1 \cup A_2$ with closed subsets A_i of X with $i = 1, 2$, we have $A_1 = X$ or $A_2 = X$. A subset Y of X is irreducible if it is irreducible as a subspace of X . An irreducible component of a topological space A is a maximal irreducible subset of X .

Both of a singleton subset and its closure in $q\text{Spec}(M)$ are irreducible. Now, applying (1) of Proposition 3.4, we obtain that

Corollary 3.9. *$D(L)$ is an irreducible closed subset of $q\text{Spec}(M)$ for every quasi-prime submodule L of M .*

Theorem 3.10. *Let M be an R -module and $Y \subseteq q\text{Spec}(M)$. Then $\mathfrak{S}(Y)$ is a quasi-prime submodule of M if and only if Y is an irreducible space.*

Proof. Let $\mathfrak{S}(Y)$ be a quasi-prime submodule of M . Let $Y \subseteq Y_1 \cup Y_2$ where Y_1 and Y_2 are two closed subsets of X . Then there are submodules N and L of M such that $Y_1 = D(N)$ and $Y_2 = D(L)$. Hence, $Y \subseteq D(N) \cup D(L) = D(N \cap L)$. By Lemma 3.1, $((N \cap L) : M) \subseteq (\mathfrak{S}(Y) : M)$. Since $(\mathfrak{S}(Y) : M)$ is a quasi-prime ideal, either $(N : M) \subseteq (\mathfrak{S}(Y) : M)$ or $(L : M) \subseteq (\mathfrak{S}(Y) : M)$. By Lemma 3.1, either $Y \subseteq D(N) = Y_1$ or $Y \subseteq D(L) = Y_2$. This yields Y is irreducible.

Assume that Y is an irreducible space. Let I and J be two ideals of R such that $I \cap J \subseteq (\mathfrak{S}(Y) : M)$. Suppose for contradiction that $I \not\subseteq (\mathfrak{S}(Y) : M)$ and $J \not\subseteq (\mathfrak{S}(Y) : M)$. Then $(IM : M) \not\subseteq (\mathfrak{S}(Y) : M)$ and $(JM : M) \not\subseteq (\mathfrak{S}(Y) : M)$. By Lemma 3.1, $Y \not\subseteq D(IM)$, $Y \not\subseteq D(JM)$. Let $P \in Y$. Then $(P : M) \supseteq (\mathfrak{S}(Y) : M) \supseteq I \cap J$. This means that either $IM \subseteq (P : M)M$ or $JM \subseteq (P : M)M$. So, by Lemma 3.1, either $D(P) \subseteq D(IM)$ or $D(P) \subseteq D(JM)$. Therefore, $Y \subseteq D(IM) \cup D(JM)$ which is a contradiction to irreducibility of Y . \square

Example 3.11. Consider $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}$ as a \mathbb{Z} -module, where p is a prime integer. It is easy to see that $L = \mathbb{Z}/p\mathbb{Z} \oplus (0)$ and $N = (0) \oplus p\mathbb{Z}$ are prime submodules of M . We have $\mathfrak{S}(q\text{Spec}(M)) \subseteq L \cap N = (0)$. Hence, $(\mathfrak{S}(q\text{Spec}(M)) : M) = ((0) : M) = (0)$ is a quasi-prime ideal of \mathbb{Z} . This implies that $\mathfrak{S}(q\text{Spec}(M))$ is a quasi-prime submodule of M . By Theorem 3.10, $q\text{Spec}(M)$ is an irreducible space.

Corollary 3.12. *Let M be an R -module and $N \leq M$.*

- (1) $V^M(N)$ is irreducible if and only if $\text{rad}(N)$ is a quasi-prime submodule.
- (2) If N is a \mathfrak{p} -primary submodule of M where $\mathfrak{p} \in \text{Max}(R)$, then $V^M(N)$ is irreducible.
- (3) Let R be a quasi-local ring. Then $\text{Max}(M)$ is irreducible.
- (4) The quasi-prime spectrum of every faithful reduced module over an integral domain is irreducible.

Proof. (1) Since $\text{rad}(N) = \mathfrak{S}(V^M(N))$, result follows immediately from Theorem 3.10. (2) Use part (1) and [Lu03, Corollary 5.7]. (3) Let \mathfrak{m} be the unique maximal ideal of R . By [Lu84, p.63, Proposition 4], $(H : M) = \mathfrak{m}$ for each $H \in \text{Max}(M)$. By Lemma 2.4(2), $\bigcap_{H \in \text{Max}(M)} H = \mathfrak{S}(\text{Max}(M))$ is a quasi-prime submodule. By Theorem 3.10, $\text{Max}(M)$ is irreducible. (4) Since M is reduced, $(\mathfrak{S}(q\text{Spec}(M)) : M) \subseteq (\mathfrak{S}(\text{Spec}(M)) : M) = (\bigcap_{P \in \text{Spec}(M)} P : M) = ((0) : M) = (0) \in \text{Spec}(R)$. The result follows from Theorem 3.10. \square

Example 3.13. (1) Let $M = \mathbb{Z} \oplus \mathbb{Z}(p^\infty)$ be a \mathbb{Z} -module. Then by Theorem 3.10, $\text{Spec}(M)$ is an irreducible space because $\mathfrak{S}(\text{Spec}(M)) = (0) \oplus \mathbb{Z}(p^\infty)$ is a prime submodule of M .

- (2) Let $M = \mathbb{Q} \oplus \mathbb{Z}/p\mathbb{Z}$ be a \mathbb{Z} -module. By Theorem 3.10, $\text{Max}(M)$ is an irreducible subset of $q\text{Spec}(M)$ because $\text{Rad}(M) = \mathbb{Q} \oplus (0)$.

Corollary 3.14. *Let M be an R -module such that $(0) \in q\text{Spec}(M)$. Then $q\text{Spec}(M)$ is an irreducible space. In particular, if R is an integral domain and M is a torsion-free R -module, then $q\text{Spec}(M)$ is an irreducible space. Moreover, $q\text{Spec}(R)$ is an irreducible space, if R is an integral domain.*

Proof. Use Theorem 3.10 and [Lu03, Lemma 4.5]. \square

Example 3.15. Consider the faithful \mathbb{Z} -module $M = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$, where p runs through the set of all prime integers. Then by Corollary 3.14, $\text{Spec}(M)$ is an irreducible space.

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a *generic point* of Y if $Y = \text{Cl}(\{y\})$. In Proposition 3.4 (1), we have seen that every element L of $q\text{Spec}(M)$ is a generic point of the irreducible closed subset $D(L)$ of $q\text{Spec}(M)$. Note that a generic point of a closed subset Y of a topological space is unique if the topological space is a T_0 -space.

Theorem 3.16. *Let M be an R -module and $Y \subseteq q\text{Spec}(M)$.*

- (1) *Then Y is an irreducible closed subset of $q\text{Spec}(M)$ if and only if $Y = D^M(L)$ for some $L \in q\text{Spec}(M)$. Thus every irreducible closed subset of $q\text{Spec}(M)$ has a generic point.*
- (2) *If M is quasi-prime-embedding, then the correspondence $D^M(L) \mapsto L$ is a bijection of the set of irreducible components of $q\text{Spec}(M)$ onto the set of minimal elements of $q\text{Spec}(M)$ with respect to inclusion.*
- (3) *Let M be a quasi-primeful R -module. Then the set of all irreducible components of $q\text{Spec}(M)$ is of the form*

$$T = \{D^M(IM) \mid I \text{ is a minimal element of } D^R(\text{Ann}(M)) \text{ w.r.t inclusion}\}.$$

- (4) *Let R be an arithmetical Laskerian ring and M be a nonzero quasi-primeful R -module. Then $q\text{Spec}(M)$ has finitely many irreducible components.*

Proof. (1) It is clear $Y = D(L)$ is an irreducible closed subset of $q\text{Spec}(M)$ for any $L \in q\text{Spec}(M)$ by Corollary 3.9. Conversely, if Y is an irreducible closed subset of $q\text{Spec}(M)$, then $Y = D(N)$ for some $N \leq M$ and $L := \mathfrak{Z}(Y) = \mathfrak{Z}(D(N)) \in q\text{Spec}(M)$ by Theorem 3.10. Hence, $Y = D(N) = D(\mathfrak{Z}(D(N))) = D(L)$ as desired.

(2) Let Y be an irreducible component of $q\text{Spec}(M)$. Since each irreducible component of $q\text{Spec}(M)$ is a maximal element of the set $\{D(N) \mid N \in q\text{Spec}(M)\}$ by (1), we have $Y = D(L)$ for some $L \in q\text{Spec}(M)$. Obviously, L is a minimal element of $q\text{Spec}(M)$, for if $T \in q\text{Spec}(M)$ with $T \subseteq L$, then $D(L) \subseteq D(T)$. So $L = T$ due to the maximality of $D(L)$ and Proposition 2.22. Let L be a minimal element of $q\text{Spec}(M)$ with $D(L) \subseteq D(N)$ for some $N \in q\text{Spec}(M)$. Then $L \in D(N)$ whence $(N : M)M \subseteq L$. By Lemma 2.4, $(N : M)M$ belongs to $q\text{Spec}(M)$. Hence, $L = (N : M)M$ due to the minimality of L . By Lemma 3.1, $D(N) = D((N : M)M) = D(L)$. This implies that $D(L)$ is an irreducible component of $q\text{Spec}(M)$, as desired.

(3) Let Y be an irreducible component of $q\text{Spec}(M)$. By part (1), $Y = D^M(L)$ for some $L \in q\text{Spec}(M)$. Hence, $Y = D^M(L) = D^M((L : M)M)$ by Lemma 3.1. So, we have $l := (L : M) \in D^R(\text{Ann}(M))$. We must show that l is a minimal element of $D^R(\text{Ann}(M))$ w.r.t inclusion. To see this let $q \in D^R(\text{Ann}(M))$ and $q \subseteq l$. Then $q/\text{Ann}(M) \in q\text{Spec}(R/\text{Ann}(M))$, and there exists an element $Q \in q\text{Spec}(M)$ such that $(Q : M) = q$ because M is quasi-primeful. So, $Y = D^M(L) \subseteq D^M(Q)$. Hence, $Y = D^M(L) = D^M(Q)$ due to the maximality of $D^M(L)$. By Proposition 3.4, we have that $l = q$. Conversely, let $Y \in T$. Then there exists a minimal element I in $D^R(\text{Ann}(M))$ such that $Y = D^M(IM)$. Since M is quasi-primeful, there exists an element $N \in q\text{Spec}(M)$ such that $(N : M) = I$.

So, $Y = D^M(IM) = D^M((N : M)M) = D^M(N)$, and so Y is irreducible by part (1). Suppose that $Y = D^M(N) \subseteq D^M(Q)$, where Q is an element of $q\text{Spec}(M)$. Since $N \in D^M(Q)$ and I is minimal, it follows that $(N : M) = (Q : M)$. Now, by Lemma 3.1,

$$Y = D^M(N) = D^M((N : M)M) = D^M((Q : M)M) = D^M(Q).$$

- (4) By assumption, the set of quasi-prime ideals are exactly the set of primary ideals (see Remark 2.2). If I is a minimal element of $D^R(\text{Ann}(M))$ and $\text{Ann}(M) = \cap_{i=1}^n Q_i$ is a minimal primary decomposition of $\text{Ann}(M)$, then $Q_i \subseteq I$ for some $1 \leq i \leq n$ (Since I is quasi-prime and $\cap_{i=1}^n Q_i \subseteq I$). By minimality of I , we get $I = Q_i$. Therefore, irreducible components of $q\text{Spec}(M)$ are the form $D^M(Q_i M)$, by part (3). \square

We introduce a base for the developed Zariski topology on $q\text{Spec}(M)$ for any R -module M . For each $a \in R$, we define $\Gamma_M(a) = q\text{Spec}(M) - D(aM)$. Then every $\Gamma_M(a)$ is an open set of $q\text{Spec}(M)$, $\Gamma_M(0) = \emptyset$, and $\Gamma_M(1) = q\text{Spec}(M)$.

Proposition 3.17. *For any R -module M , the set $B = \{\Gamma_M(a) \mid a \in R\}$ forms a base for the developed Zariski topology on $q\text{Spec}(M)$.*

Proof. We may assume that $q\text{Spec}(M) \neq \emptyset$. Let U be any open subset in $q\text{Spec}(M)$. There exists a submodule N of M such that

$$\begin{aligned} U &= q\text{Spec}(M) - D(N) = q\text{Spec}(M) - D((N : M)M) \\ &= q\text{Spec}(M) - D\left(\sum_{a_i \in (N:M)} a_i M\right) \\ &= q\text{Spec}(M) - D\left(\sum_{a_i \in (N:M)} (a_i M : M)M\right) \\ &= q\text{Spec}(M) - \bigcap_{a_i \in (N:M)} D(a_i M) \\ &= \bigcup_{a_i \in (N:M)} \Gamma_M(a_i). \end{aligned}$$

\square

Proposition 3.18. *Let M be an R -module, $a \in R$ and $\psi : q\text{Spec}(M) \rightarrow q\text{Spec}(R/\text{Ann}(M))$ be the natural map of $q\text{Spec}(M)$.*

- (1) $\psi^{-1}(\Gamma_{\bar{R}}(\bar{a})) = \Gamma_M(a)$;
- (2) $\psi(\Gamma_M(a)) \subseteq \Gamma_{\bar{R}}(\bar{a})$. If M is quasi-primeful, then $\psi(\Gamma_M(a)) = \Gamma_{\bar{R}}(\bar{a})$;
- (3) If M is quasi-primeful, then $q\text{Spec}(M)$ is a compact space.
- (4) If M is finitely generated multiplication, then $q\text{Spec}(M)$ is compact.

Proof. (1) By Proposition 3.2, we have

$$\begin{aligned} \psi^{-1}(\Gamma_{\bar{R}}(\bar{a})) &= \psi^{-1}(q\text{Spec}(\bar{R}) - D(\bar{a}\bar{R})) \\ &= q\text{Spec}(M) - \psi^{-1}(D(\bar{a}\bar{R})) \\ &= q\text{Spec}(M) - D(aM) = \Gamma_M(a). \end{aligned}$$

- (2) This follows from (1).

- (3) By Proposition 3.17, the set $B = \{\Gamma_M(a) \mid a \in R\}$ is a base for the developed Zariski topology on $q\text{Spec}(M)$. For any open cover of $q\text{Spec}(M)$, there is a family $\{a_\lambda \in R \mid \lambda \in \Lambda\}$ of elements of R such that $q\text{Spec}(M) = \bigcup_{\lambda \in \Lambda} \Gamma_M(a_\lambda)$ and for each $\lambda \in \Lambda$, there is an open set in the covering containing $\Gamma_M(a_\lambda)$. By part (2),

$$\begin{aligned} q\text{Spec}(\bar{R}) &= \Gamma_{\bar{R}}(\bar{1}) \\ &= \psi(\Gamma_M(1)) \\ &= \psi(q\text{Spec}(M)) \subseteq \bigcup_{\lambda \in \Lambda} \psi(\Gamma_M(a_\lambda)) \\ &= \bigcup_{\lambda \in \Lambda} \Gamma_{\bar{R}}(\bar{a}_\lambda). \end{aligned}$$

By [Azi08, Theorem 4.1], $q\text{Spec}(\bar{R})$ is compact, hence there exists a finite subset Λ' of Λ such that $q\text{Spec}(\bar{R}) \subseteq \bigcup_{\lambda \in \Lambda'} \Gamma_{\bar{R}}(\bar{a}_\lambda)$. By part (1),

$$\begin{aligned} q\text{Spec}(M) &= \Gamma_M(1) \\ &= \psi^{-1}(\Gamma_{\bar{R}}(\bar{1})) \\ &= \psi^{-1}(q\text{Spec}(\bar{R})) \subseteq \bigcup_{\lambda \in \Lambda'} \psi^{-1}(\Gamma_{\bar{R}}(\bar{a}_\lambda)) \\ &= \bigcup_{\lambda \in \Lambda'} \Gamma_M(a_\lambda). \end{aligned}$$

- (4) Let $\{D^M(N_\lambda)\}_{\lambda \in \Lambda}$ be an arbitrary family of closed subsets of $q\text{Spec}(M)$, where $N_\lambda \leq M$ for each $\lambda \in \Lambda$ such that $\bigcap_{\lambda \in \Lambda} D^M(N_\lambda) = \emptyset$. Hence, we have $D^M(\sum_{\lambda \in \Lambda} (N_\lambda : M)M) = \emptyset$. Since M is multiplication, $\Omega^M(\sum_{\lambda \in \Lambda} (N_\lambda : M)M) = \emptyset$, so $M = \sum_{\lambda \in \Lambda} (N_\lambda : M)M$. Since M is finitely generated, there exists a finite subset Λ' of Λ such that $M = \sum_{\lambda \in \Lambda'} (N_\lambda : M)M$. This completes the proof. \square

A topological space X is said to be *Noetherian* if the open subsets of X satisfy the ascending chain condition. Since closed subsets are complements of open subsets, it comes to the same thing to say that the closed subsets of X satisfy the descending chain condition.

Theorem 3.19. *Let M be an R -module.*

- (1) *If M satisfies ACC on quasi-semiprime submodules, then $q\text{Spec}(M)$ is a Noetherian topological space. In particular, quasi-prime spectrum of every Noetherian module is a Noetherian topological space (see [Azi08, Theorem 4.2]).*
- (2) *If for every submodule N of M there exists a finitely generated submodule L of N such that $\mathfrak{S}(\Omega^M(N)) = \mathfrak{S}(\Omega^M(L))$, then $q\text{Spec}(M)$ is a Noetherian topological space.*
- (3) *If R satisfies ACC on quasi-semiprime ideals, then $q\text{Spec}(M)$ is a Noetherian topological space. In particular, for every module M over a Noetherian ring, $q\text{Spec}(M)$ is a Noetherian topological space.*

Proof. (1) Let $D(N_1) \supseteq D(N_2) \supseteq \dots$ be a descending chain of closed subsets of $q\text{Spec}(M)$. We have an ascending chain of quasi-semiprime submodules

of M , $\mathfrak{Z}(D(N_1)) \subseteq \mathfrak{Z}(D(N_2)) \subseteq \dots$ which is stationary by assumption. So, there exists a positive integer k such that $\mathfrak{Z}(D(N_k)) = \mathfrak{Z}(D(N_{k+i}))$, for each $i = 1, 2, \dots$. By Proposition 3.4, $D(N_k) = D(N_{k+i})$, and so $q\text{Spec}(M)$ is a Noetherian topological space.

- (2) Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$ be an ascending chain of quasi-semiprime submodules of M , and let $N = \cup_i N_i$. By assumption, there exists a finitely generated submodule L of N such that $\bigcap_{P \in \Omega^M(N)} P = \bigcap_{Q \in \Omega^M(L)} Q$. Hence there exists a positive integer n such that $L \subseteq N_n$. Then

$$\bigcap_{P \in \Omega^M(N)} P = \bigcap_{Q \in \Omega^M(L)} Q \subseteq N_n \subseteq N \subseteq \bigcap_{P \in \Omega^M(N)} P,$$

so that $N_n = N_{n+1} = N_{n+2} = \dots$. Hence, M satisfies *ACC* on quasi-semiprime submodules. By (1), $q\text{Spec}(M)$ is a Noetherian topological space.

- (3) Let $D(N_1) \supseteq D(N_2) \supseteq \dots$ be a descending chain of closed subsets of $q\text{Spec}(M)$. By assumption, there exists a positive integer k such that $(\mathfrak{Z}(D(N_k)) : M)M = (\mathfrak{Z}(D(N_{k+i})) : M)M$, for each $i = 1, 2, \dots$. By Lemma 3.1, $D(\mathfrak{Z}(D(N_k))) = D(\mathfrak{Z}(D(N_{k+i})))$. By Proposition 3.4, $D(N_k) = D(N_{k+i})$, and so $q\text{Spec}(M)$ is a Noetherian space. \square

Remark 3.20. Let X be a Noetherian topological space. Then every subspace of X is compact. In particular, X is compact (see [AM69, p. 79, Ex. 5]).

As a consequence of Remark 3.20, we have

Corollary 3.21. *For an R -module M , $q\text{Spec}(M)$ is a compact space in each of the following cases.*

- (1) M satisfies *ACC* on quasi-semiprime submodules;
- (2) R satisfies *ACC* on quasi-semiprime ideals.

For example, quasi-prime spectrum of every \mathbb{Z} -module is compact space.

Proposition 3.22. *Let M be a quasi-prime-embedding R -module. If $q\text{Spec}(M)$ is a Noetherian space, then*

- (1) *Every ascending chain of quasi-prime submodules of M is stationary;*
- (2) *$q\text{Spec}(M)$ has finitely many minimal element. In particular, every multiplication module over a Noetherian ring has finitely many minimal quasi-prime submodules.*

Proof. (1) Let $N_1 \subseteq N_2 \subseteq \dots$ be an ascending chain of quasi-prime submodules of M . Then $D(N_1) \supseteq D(N_2) \supseteq \dots$ is a descending chain of closed subsets of $q\text{Spec}(M)$, which is stationary by assumption. There exists an integer $k \in \mathbb{N}$ such that $D(N_k) = D(N_{k+i})$ for each $i \in \mathbb{N}$. By Proposition 2.22, we have $N_k = N_{k+i}$ for each $i \in \mathbb{N}$. This completes the proof. (2) Since every Noetherian topological space has finitely many irreducible components, the result follows from Theorem 3.16(2). For last statement, use Corollary 2.23 and Theorem 3.19. \square

We recall that if X is a finite space, then X is a T_1 if and only if X is the discrete space. We also recall that a topological space is called Hausdorff if any two distinct points possess disjoint neighborhoods. So, we have the following corollary.

Corollary 3.23. *Let R be a Noetherian ring and M be a finitely generated R -module. Then the following statements are equivalent:*

- (1) $q\text{Spec}(M)$ is a Hausdorff space;
- (2) $q\text{Spec}(M)$ is a T_1 -space;
- (3) $q\text{Spec}(M)$ is a discrete space;
- (4) M is a multiplication module and $q\text{Spec}(M) = \text{Max}(M)$.

Proof. (1) \Rightarrow (2) and (3) \Rightarrow (1) are clear. (2) \Leftrightarrow (4) follows from Theorem 3.7. (2) \Rightarrow (3) By Proposition 3.22, M has finitely many minimal quasi-prime submodules. By Theorem 3.7, $q\text{Spec}(M)$ is finite. Therefore, $q\text{Spec}(M)$ is a discrete space. \square

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